Triple Integrals

Triple integrals in Rectangular Coordinates

Volume $= \iiint_D dV$, where D is the bounded region.

Finding limits of integration in order $dz\,dy\,dx$
order matters.

$$\iiint_D F(x,y,z)\,dz\,dy\,dx$$

Master plan for triple integrals

0. Identify the region E & the function $f(x,y,z)$
1. Draw E in xyz-space $(\mathbb{R}^3)$
2. Decide the order of $x, y, \& z$.
3. First variable using constants only.
4. Second variable using constants or expressions in the first variable only.

   Note: If x & y are your first two variables, they should describe the shadow on the xy-plane just like a double integral. If you pick other variables, describe the shadow on the corresponding wall.

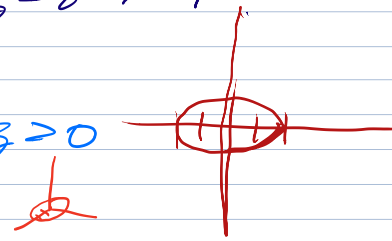
5. Third variable constant or expression in the first two.

- $$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy = \frac{1}{3}\ln 2$$

Ex) Find the volume of the region D enclosed by the surface $z=x^2+3y^2$ + $z=8-x^2-y^2$

Ans: Find the intersection

$x^2+3y^2=8-x^2-y^2$, $z>0$



$2x^2+4y^2=8$ $\quad \frac{x^2}{4}+\frac{y^2}{2}=1$

$x^2+2y^2=4$ The boundary of the region R

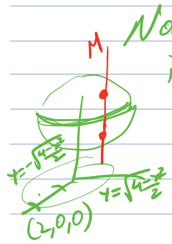
(the projection of D onto the xy plane)

is an ellipse with the same equation.

$x^2+2y^2=4$ The upper boundary $y=\sqrt{\frac{4-x^2}{2}}$

The lower boundary $y=-\sqrt{\frac{4-x^2}{2}}$

Now we have to find the z-limits of integration.



M passing through a typical point (x,y) in R parallel to the z axis.

Centers D at $z=x^2+3y^2$ +

leaves $z=8-x^2-y^2$

Finally we find the x limits of integration

from $x=-2 \Rightarrow x=2$

+ y limit $y=-\sqrt{\frac{4-x^2}{2}}$ to $y=\sqrt{\frac{4-x^2}{2}}$

z limit $z=x^2+3y^2$ to $z=8-x^2-y^2$

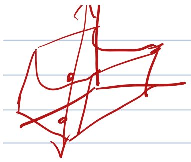
$$V=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx = 8\pi\sqrt{2}$$

Ex) Find the volume

The region between the cylinder

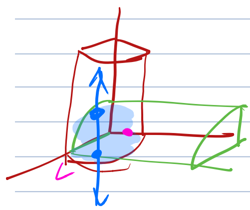
$z=y^2$ + the xy plane that is bounded

by plane $x=0$, $x=1$, $y=-1$, $y=1$.

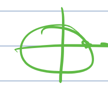


$$\int_0^1 \int_{-1}^1 \int_0^{y^2} dz\,dy\,dx = 2/3$$

The region common to the interior of the cylinder $x^2+y^2=1$ & $x^2+z^2=1$, 1/8 of which shown in the figure



$$8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx$$



$y=\sqrt{1-x^2}$

$z=\sqrt{1-x^2}$

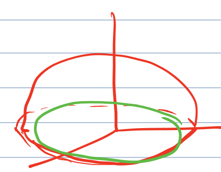
$$= \frac{16}{3}$$

Ex) Top half of the solid unit ball.

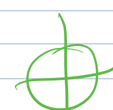
$x^2+y^2+z^2=1$

$x^2+y^2=1$

$-1,\ 1$



$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$



$x^2+y^2=1$

$y=\pm\sqrt{1-x^2}$

$z=\pm\sqrt{1-x^2-y^2}$

$$= \frac{2\pi}{3}$$